Define $L^1_{\text{bc}}(\mathbb{R})$ to be the set of functions $f : \mathbb{R} \to \mathbb{C}$ which are continuous, bounded, and satisfy

$$\|f\|_1 = \int_{-\infty}^{\infty} |f| < \infty$$

Then given $f \in L^1_{\text{bc}}(\mathbb{R})$, define its Fourier transform $\hat{f}$ to be

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} dx$$

Claim Let $g(x) = -2\pi ixf(x)$, and suppose that $g \in L^1_{\text{bc}}(\mathbb{R})$. Then $\hat{f} \in C^1(\mathbb{R}, \mathbb{C})$, i.e. differentiable with continuous derivative, and $\hat{f}' = \hat{g}$.

A few important lemmas:

Lemma Let $x, y \in \mathbb{R}$. Then $|e^{ix} - e^{iy}| \leq |x - y|$. In particular, $|e^{ix} - 1| \leq |x|$.

Proof. Suppose $x \leq y$. Then,

$$\begin{aligned} |e^{ix} - e^{iy}| &= \left|-i\, e^{it}\big|_x^y\right| \\ &= \left|\int_x^y e^{it}\right| \\ &\leq \int_x^y 1 \\ &= y - x \\ &= |x - y| \end{aligned}$$

If $x \geq y$, then simply do $|e^{ix} - e^{iy}| = |e^{iy} - e^{ix}|$ first to obtain $x - y = |x - y|$. Choosing $y = 0$ gives the particular case. ■

Lemma Let $f : V \to W$ for some normed linear spaces V, W. Let $c \in V$.

$$\lim_{x \to c} f(x) = f(c)$$

iff $\forall (c_n) \subseteq V - \{c\}$ such that $c_n \to c$, $\lim f(c_n) = f(c)$.

Proof. The first direction is just definitions. Conversely, suppose $\forall (c_n) \subseteq V - \{c\}$ such that $c_n \to c$, $\lim f(c_n) = f(c)$. Suppose for a contradiction that $\lim_{x \to c} f(x) \neq f(c)$, i.e. $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x \in V$ satisfying $\|x - c\| < \delta$ and $\|f(x) - f(c)\| \geq \varepsilon$. This generates a sequence $(c_n) \subseteq V - \{c\}$ (since $x = c$ clearly doesn't fail) such that $\forall n$, $\|c_n - c\| < 1/n$ and $\|f(c_n) - f(c)\| \geq \varepsilon$. Then $c_n \to c$, but $f(c_n) \nrightarrow f(c)$. This contradicts the hypothesis. ■

Lemma Let $f \in L^1_{\text{bc}}(\mathbb{R})$. Then $\hat{f}$ is continuous.

Proof. Notice if $\|f\|_1 = 0$, then $f = 0$, in which case the lemma is trivial. Now let $y_0 \in \mathbb{R}$. Let $\varepsilon > 0$. Note that

$$\begin{aligned}\left|\hat{f}(y) - \hat{f}(y_0)\right| &= \left|\int_{-\infty}^{\infty} f(x)\left(e^{-2\pi ixy} - e^{-2\pi ixy_0}\right)dx\right| \\ &= \left|\int_{-\infty}^{\infty} f(x)\, e^{-2\pi ixy_0}\left(e^{-2\pi ix(y-y_0)} - 1\right)dx\right| \\ &\leq \int_{-\infty}^{\infty} |f(x)|\left|e^{-2\pi ix(y-y_0)} - 1\right| dx\end{aligned}$$

Since $\|f\|_1 < \infty$, therefore $\exists R > 0$ such that

$$\int_{|x|>R} |f| < \frac{\varepsilon}{4}$$

Hence, since

$$\int_{-\infty}^{\infty} |f(x)|\left|e^{-2\pi ix(y-y_0)} - 1\right| dx \leq 2\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

therefore

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|\left|e^{-2\pi ix(y-y_0)} - 1\right| dx &= \int_{|x|\leq R} |f(x)|\left|e^{-2\pi ix(y-y_0)} - 1\right| dx + \\ &\quad + \int_{|x|>R} |f(x)|\left|e^{-2\pi ix(y-y_0)} - 1\right| dx \\ &< 2\pi|y-y_0|\int_{|x|\leq R} |x|\,|f(x)|\, dx + 2\int_{|x|>R} |f(x)|\, dx \\ &< 2R\pi|y-y_0|\int_{|x|\leq R} |f(x)|\, dx + \frac{\varepsilon}{2} \\ &\leq 2R\pi|y-y_0|\,\|f\|_1 + \frac{\varepsilon}{2}\end{aligned}$$

Thus choosing $|y - y_0| < \varepsilon / (R\pi \|f\|_1)$ shows the claim. ∎

Also recall a weak version of the dominated convergence theorem:

Dominated convergence theorem (Weak) Let $f_n : \mathbb{R} \to \mathbb{C}$ continuous which converge locally uniformly to f. If $\exists g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\int_{-\infty}^{\infty} g < \infty$ and $|f_n| \leq g$, then

$$\lim \int_{-\infty}^{\infty} f_n = \int_{-\infty}^{\infty} f$$

Now for the claim:

Proof of Claim. Note

$$\begin{aligned} \hat{g}(y) &= \int_{-\infty}^{\infty} g(x) e^{-2\pi ixy} dx \\ &= -2\pi i \int_{-\infty}^{\infty} x f(x) e^{-2\pi ixy} dx \end{aligned}$$

Recall $\hat{f} : \mathbb{R} \to \mathbb{C}$. So we can differentiate. Note that

$$\begin{aligned} \frac{\hat{f}(y+h) - \hat{f}(y)}{h} &= \frac{1}{h} \int_{-\infty}^{\infty} f(x) \left(e^{-2\pi ix(y+h)} - e^{-2\pi ixy}\right) dx \\ &= \frac{1}{h} \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \left(e^{-2\pi ixh} - 1\right) dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \varphi(x,h)\, dx \end{aligned}$$

where

$$\varphi(x,h) = \frac{e^{-2\pi ixh} - 1}{h}$$

Hence

$$\hat{f}'(y) = \lim_{h \to 0} \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \varphi(x,h)\, dx$$

By a lemma it is equivalent to show

$$\hat{f}'(y) = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \varphi(x,h_n)\, dx$$

for an arbitrary $(h_n) \subseteq \mathbb{R} - \{0\}$ that converges to 0, thereby realizing $\varphi(x,h_n)$ as a sequence of continuous functions in x. It is now enough to show that

1. $\varphi(x,h_n)$ converge locally uniformly to $-2\pi ix$
2. $\exists k : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\int_{-\infty}^{\infty} k < \infty$ and $|f(x) e^{-2\pi ixy} \varphi(x,h_n)| \leq k$

by the above weaker version of the dominated convergence theorem, because it would then follow that $f(x) e^{-2\pi ixy} \varphi(x,h_n)$ converge locally uniformly to $-2\pi ixf(x) e^{-2\pi ixy}$, and so

$$\lim \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \varphi(x, h_n)\, dx = -2\pi i \int_{-\infty}^{\infty} x f(x) e^{-2\pi ixy} dx = \hat{g}(y)$$

as calculated earlier, where $\hat{g}(y)$ is continuous by a lemma since $g \in L^1_{\mathrm{bc}}(\mathbb{R})$. To see this additional local uniform convergence, recall that $\exists M > 0$ such that $|f| \leq M$, so that

$$\left| f(x) e^{-2\pi ixy} \varphi(x, h_n) + 2\pi ix f(x) e^{-2\pi ixy} \right| \leq M \cdot 1 \cdot |\varphi(x, h_n) + 2\pi ix|$$

and then the convergence follows from that of $\varphi(x, h_n)$.

Consider (1). Let $x \in \mathbb{R}$, and let $\varepsilon > 0$. Note that

$$\begin{aligned}
\left| \frac{e^{-2\pi ixh_n} - 1}{h_n} + 2\pi ix \right| &= \frac{1}{|h_n|} \left| e^{-2\pi ixh_n} - 1 + 2\pi ixh_n \right| \\
&= \frac{1}{|h_n|} \left| \sum_{j=2}^{\infty} \frac{(-2\pi ixh_n)^j}{j!} \right| \\
&= \frac{1}{|h_n|} \left| (-2\pi ixh_n)^2 \sum_{j=2}^{\infty} \frac{(-2\pi ixh_n)^{j-2}}{j!} \right| \\
&= 4\pi^2 |x|^2 |h_n| \left| \sum_{j=0}^{\infty} \frac{(-2\pi ixh_n)^j}{(j+2)!} \right| \\
&\leq 4\pi^2 |x|^2 |h_n| \sum_{j=0}^{\infty} \frac{|-2\pi ixh_n|^j}{j!} \\
&= 4\pi^2 |x|^2 |h_n| e^{2\pi |x| |h_n|}
\end{aligned}$$

Since $\lim h_n = 0$, therefore $\lim e^{2\pi |x||h_n|} = 1$. Hence $\exists N_1$ such that $\forall n \geq N_1$,

$$\left| e^{2\pi |x||h_n|} - 1 \right| < 1 \Longrightarrow e^{2\pi |x||h_n|} < 2$$

Hence for $n \geq N_1$,

$$4\pi^2 |x|^2 |h_n| e^{2\pi |x||h_n|} < 8\pi^2 |x|^2 |h_n|$$

Choose any neighbourhood $x \in (-M, M)$ for $M > 0$, so that

$$8\pi^2 |x|^2 |h_n| < 8\pi^2 M^2 |h_n|$$

Now $\exists N_2$ such that $\forall n \geq N_2$, $|h_n| < \varepsilon / (8\pi^2 M^2)$. Hence $\forall n \geq \max\{N_1, N_2\}$, $|\varphi(x, h_n) + 2\pi ix| < \varepsilon$.

For (2), let $k = |g| = 2\pi |x| |f(x)|$, since

$$\left| f(x) e^{-2\pi ixy} \varphi(x, h_n) \right| \leq |f(x)| \, 2\pi |x|$$

because $\left| e^{-2\pi ixh_n} - 1 \right| \leq |-2\pi x h_n|$ by a lemma, and $\|g\|_1 < \infty$ by assumption that $g \in L^1_{\mathrm{bc}}(\mathbb{R})$. ■