**Claim** Let $\mu : P(\mathbb{Z}^+) \to [0,\infty]$ be the counting measure on $\mathbb{Z}^+$ and let $f : \mathbb{Z}^+ \to [0,\infty]$. Then

$$\int_{\mathbb{Z}^+} f d\mu = \sum_{k=1}^{\infty} f(k)$$

*Proof.* First consider when $\exists k_0$ such that $f(k_0) = \infty$. In this case define the simple functions $f_n : \mathbb{Z}^+ \to [0,\infty)$, $k \mapsto n\chi_{\{k_0\}}$. Then $0 \le f_n \le f$. By definition of integration of a simple function,

$$\int_{\mathbb{Z}^+} f_n d\mu = n\mu\left(\{k_0\} \cap \mathbb{Z}^+\right) = n$$

Then $\int_{\mathbb{Z}^+} f_n d\mu$ is not bounded above and hence by definition of the Lebesgue integral,

$$\int_{\mathbb{Z}^+} f d\mu = \sup \int_{\mathbb{Z}^+} s d\mu = \infty = \sum_{k=1}^{\infty} f(k)$$

where $s$ is any simple function such that $0 \le s \le f$.

Now suppose that $f$ is finite-valued. Enumerate the distinct values of $f$ according to $f(1)$, $f(2)$, $\ldots$: $y_1$, $y_2$, $\ldots$ (e.g. $f^{-1}(\{y_1\}) = \{1\}$, $f^{-1}(\{y_2\}) = \{2,3,4\}$, etc.). Define the simple functions

$$f_n = \sum_{k=1}^{n} y_k \chi_{f^{-1}(\{y_k\})}$$

Then since $f_n(i) = y_k$ iff $f(i) = y_k$, therefore $f_n(i) = f(i)$ for $1 \le i \le \sup f^{-1}(\{y_n\})$ and beyond that $f_n \equiv 0$. Hence $0 \le f_n \le f$. Also

$$\int_{\mathbb{Z}^+} f_n d\mu = \sum_{k=1}^{n} y_k \mu\left(f^{-1}(\{y_k\})\right)$$

But by definition of the $y_k$ and the fact that $\mu(f^{-1}(\{y_k\}))$ counts the number of "duplicates" of $y_k$, really

$$\sum_{k=1}^{n} y_k \mu\left(f^{-1}(\{y_k\})\right) = \sum_{k=1}^{\sup f^{-1}(\{y_n\})} f(k)$$

Consequently $\int_{\mathbb{Z}^+} f_n d\mu$ is increasing and limits to $\sum_{k=1}^{\infty} f(k)$. By the monotone convergence theorem for sequences and the definition of the Lebesgue integral the claim then follows. ■